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DISCONTINUOUS VELOCITY FIELDS IN A HARDENING RIGID-PLASTIC MATERIAL

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The proposed model and the properties obtained on the surface of velocity discontinuity are used to study two processes of plastic deformation: lateral extrusion and torsion of a hollow disk. In both cases, analytical solutions are obtained. The limits of applicability of the solution to lateral extrusion are determined.

In most cases, an approximate analysis of the technological processes of plastic metal working is based on the introduction of discontinuous velocity fields. When widespread material models are used, discontinuous velocity fields are admissible only for an ideal rigid-plastic material in a rigorous formulation. However, these fields are used under the conditions of creep [1], viscoplasticity [2], and plasticity with allowance for work hardening [3] in an approximate formulation. Moreover, few experiments have shown that in real materials, velocity discontinuities exist without violating the continuity [4] (the flow pattern that can be interpreted as a flow with velocity discontinuity with an accuracy sufficient for most practical applications is observed).

In the present paper, we show that under certain restrictions, in a rigorous formulation the velocity discontinuities are possible within the framework of the hardening rigid-plastic material model for hardening the curves that correspond to the real properties of materials. For some materials such as mild low-carbon steel, the hardening curve has a yielding site [5]. In this case, bounded velocity discontinuities can occur if the material point intersects the velocity-discontinuity surface when it moves from the undeformed (or insignificantly deformed) region. The limitations on the velocity jump depend on the dimension of the yielding site. At the same time, the hardening law proposed by Voce [6] is supported by experiments on many materials subjected to the developed plastic strains [7–9]. The corresponding idealization of the hardening curve makes it possible to introduce the velocity discontinuities for plastic strains that occur in the traditional processes of plastic metal working. In constructing the idealized hardening curve, the specific plastic work or accumulated equivalent plastic strain is used as a hardening parameter. For a material subject to the Mises yield condition, these hypotheses are equivalent for a continuous velocity field (see, e.g., [10]). However, if the plastic work is used as a hardening parameter, the equation of the hardening law can be written in divergent form [11], which can be used to study the properties of the generalized solutions. If, however, the equivalent strain is assumed to be the hardening parameter, the generalized solutions can be investigated by the method proposed by Hill [12].

1. Hardening Curve and Velocity Discontinuities. At the initial stage of plastic flow, the hardening curve of many metals has a yielding site. Thus, the plastic flow in uniaxial tension occurs for the stress $\sigma = \sigma_0 = \text{const}$. This state exists until the hardening parameter reaches a certain value at which σ becomes dependent on this parameter. Upon moderate strains, this dependence is described by various functions, which are reviewed in [7]. However, for large strains, the hardening curves of many materials tend asymptotically to a line parallel to the abscissa axis where the hardening parameter is laid off. Voce [6]

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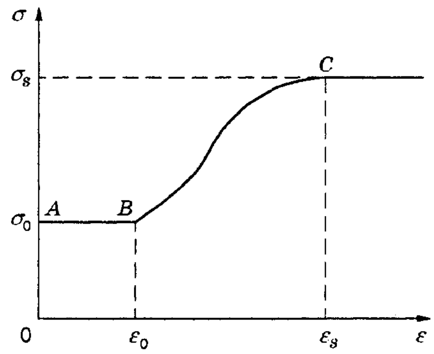

Fig. 1

obtained an analytical solution for this hardening curve, and Sevillano et al. [7] reviewed other models. For materials with a face-centered cubic lattice, the hardening curve becomes almost asymptotic for the strain $\varepsilon \approx 3$ [7].

Thus, within the accuracy sufficient for most practical applications, the hardening curve can be taken in the form shown in Fig. 1. The segment AB corresponds to the yielding site with $\sigma = \sigma_0$. The position of the point B is determined by the strain ε_0. The segment BC corresponds to hardening for an arbitrary relation between σ and ε. The asymptote to which the real hardening curve tends is determined by the equation $\sigma = \sigma_s$. It is assumed that the idealized hardening curve joins the asymptote at the point C determined by the strain ε_s. As was mentioned above, we have $\varepsilon_s \approx 3$ for some materials. To generalize the hardening curve to a complex stress state, we assume that the Mises yield criterion

$$\sqrt{3/2}\,(s_{ij}s_{ij})^{1/2} = \sigma_e \tag{1.1}$$

is valid. Here s_{ij} are the components of the deviatoric stress and σ_e is a known function of the hardening parameter. The form of the function σ_e is defined by the hardening curve upon uniaxial tension (Fig. 1). Thus, this function is represented by various analytical expressions, depending on the value of the hardening parameter. If the accumulated plastic strain $\varepsilon_{\rm eq}$ is assumed to be the hardening parameter, then

$$\sigma_e = \sigma_0 \quad \text{for} \quad \varepsilon_{\rm eq} \leqslant \varepsilon_0; \tag{1.2}$$

$$\sigma_e = \sigma_s \quad \text{for} \quad \varepsilon_{\rm eq} \geqslant \varepsilon_s. \tag{1.3}$$

When $\varepsilon_0 < \varepsilon_{\rm eq} < \varepsilon_s$, the function $\sigma_e(\varepsilon_{\rm eq})$ can be taken in the form of one of known relations [7]. The accumulated plastic strain is determined by the equation

$$\frac{d\varepsilon_{eq}}{dt} = \sqrt{\frac{2}{3}}\,(\xi_{ij}\xi_{ij})^{1/2}. \tag{1.4}$$

Here t is the time and ξ_{ij} are the components of the strain-rate tensor. The plastic work W can be used as a hardening parameter as well. In this case, W_0 and W_s play, respectively, the roles of the quantities ε_0 and ε_s, which determine the boundaries between the qualitatively different laws of material behavior.

Solutions with discontinuous velocity fields are possible if condition (1.2) or (1.3) holds. In this case, the general relations on the velocity-discontinuity surface are the same as in the ideal plasticity and are determined, for example, in [5, 11]. However, in the model proposed, additional restrictions are imposed on these relations. The fulfillment of condition (1.2) or (1.3) on the discontinuity surface is primarily a restriction itself. When condition (1.3) holds, there are no other restrictions. If condition (1.2) holds, there is one more restriction on the magnitude of the velocity jump. With the plastic work used as a hardening parameter, we obtain [11]

$$v_n[W] = [v_\tau]\sigma_0/\sqrt{3} \tag{1.5}$$

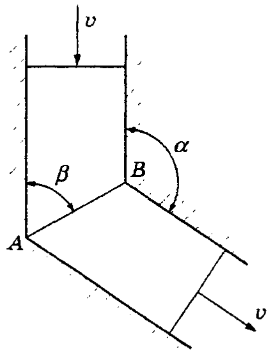

Fig. 2

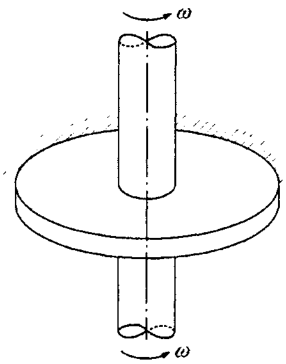

Fig. 3

on the discontinuity surface. Here the brackets denote the jump in the function and v_n and v_τ are, respectively, the projections of the velocity onto the normal and tangent to the discontinuity surface. Let the quantity W be equal to W_n at the material point just before it intersects the discontinuity surface. Evidently, for satisfaction of condition (1.2), the quantity W_n must satisfy the inequality $0 \leqslant W_n \leqslant W_0$. From (1.5), we find that the velocity jump is bounded by the condition

$$0 \leqslant [v_\tau] \leqslant \sqrt{3}\, v_n (W_0 - W_n)/\sigma_0. \tag{1.6}$$

If the accumulated plastic strain is used as a hardening parameter, the relation on the velocity-discontinuity surface can be found from (1.4) by Hill's method [12]: $[\varepsilon_{\rm eq}] = [v_\tau]/(\sqrt{3}\, v_n)$. In this case, an inequality similar to (1.6) takes the form

$$0 \leqslant [v_\tau] \leqslant \sqrt{3}\, v_n (\varepsilon_0 - \varepsilon_n), \tag{1.7}$$

where ε_n is the value of $\varepsilon_{\rm eq}$ at the material point just before it intersects the velocity-discontinuity surface and $0 \leqslant \varepsilon_n \leqslant \varepsilon_0$.

Thus, the velocity discontinuities in hardening materials determined by the hardening curve in Fig. 1 can occur in stress states that satisfy (1.2) or (1.3) and when restriction (1.6) or (1.7) holds. We note that the case where these restrictions are valid is characteristic of many steady processes of plastic metal working such as rolling and extrusion. To model these processes, the velocity-discontinuity surfaces are introduced when the material enters the deformation site where condition (1.2) holds and at the moment when the material leaves this site and the fulfillment of condition (1.3) is probable. Whether or not inequality (1.6) or (1.7) is satisfied is determined by the parameters of a particular process.

2. Lateral Extrusion. We consider lateral extrusion (Fig. 2), which has recently been in widespread use for improving the material properties [4, 13–15]. The simplest solution can be obtained under the assumption that the strain is concentrated on the velocity-discontinuity surface (plane) AB. Without loss of generality, one can assume that the velocity v of the translationally moving rigid zones is equal to unity. The cross section of an article can be arbitrary. The continuity condition of the normal velocity component implies that $\beta = \alpha/2$. Hence, the velocity jump is

$$[v_\tau] = 2\cos(\alpha/2). \tag{2.1}$$

Let the initial blank be in an undeformed state: $\varepsilon_n = 0$. We choose $\varepsilon_{\rm eq}$ as a hardening parameter. It follows from (1.7) and (2.1) that

$$\tan(\alpha/2) \geqslant (2/\sqrt{3})\varepsilon_0^{-1}. \tag{2.2}$$

For this material, the inequality (2.2) determines the maximum angle α at which the above solution is valid.

3. Torsion of a Hollow Disk. We consider the deformation of a hollow disk pierced by a rod that rotates at a constant angular velocity ω (Fig. 3). The external edge of the disk is clamped; the attachment condition is specified on the contact surface of the disk and rod. The strain in the disk is assumed to be plane. There is no solution of this problem for an ideal rigid-plastic material [16]; to be precise, there exists a solution according to which the disk is rigid, whereas the strain is concentrated on the velocity-discontinuity surface that coincide with the contact surface. With the inequality $d\sigma_e/d\varepsilon_{\rm eq} > 0$ satisfied for all $\varepsilon_{\rm eq}$, the solution for a hardening rigid-plastic material was obtained by Collins [17] and that for a powdered material was given by Lippmann [18].

We introduce the cylindrical coordinates $(r\theta z)$ with the z axis coinciding with the symmetry axis of the rod, whose radius is denoted by R. In accordance with the results of [16], a solution that corresponds to condition (1.2) must degenerate into the velocity-discontinuity surface $r = r_d$; moreover, as follows from (1.7), the condition $[v_\tau] = \sqrt{3}\, v_n \varepsilon_0$ must hold on this surface for $\varepsilon_n = 0$. Since the disk is immovable for $r \geqslant r_d$, we have $[v_\tau] = |v_\theta|$, where v_θ is calculated for $r = r_d$ in the deformable region of the disk. Thus,

$$|v_\theta| = \sqrt{3}\, v_n \varepsilon_0 \quad \text{for} \quad r = r_d. \tag{3.1}$$

Furthermore,

$$v_n = \frac{dr_d}{dt}. \tag{3.2}$$

In the deformable region of the disk, σ_e in (1.1) corresponds to the section BC on the hardening curve (see Fig. 1). Assuming that only the stress-tensor component $\tau_{r\theta}$ differs from zero, we obtain

$$\tau_{r\theta} = \sigma_e/\sqrt{3}. \tag{3.3}$$

It follows from the associated flow law that only the strain-rate tensor component $\xi_{r\theta}$ is nonzero. Setting $v_r = 0$ and $v_\theta = v(r, t)$, we obtain

$$2\xi_{r\theta} = \frac{\partial v}{\partial r} - \frac{v}{r}. \tag{3.4}$$

We find from (1.4) and (3.4) that

$$\frac{\partial \varepsilon_{\rm eq}}{\partial t} = \frac{r}{\sqrt{3}} \frac{\partial (v/r)}{\partial r}. \tag{3.5}$$

The only nontrivial equation of equilibrium has the form $\partial\tau_{r\theta}/\partial r + 2\tau_{r\theta}/r = 0$. It has the general solution

$$\tau_{r\theta} = \tau_0 r^{-2}. \tag{3.6}$$

The time function τ_0 is determined from the condition $\tau_{r\theta} = \sigma_0/\sqrt{3}$ for $r = r_d$. Solution (3.6) can now be rewritten in the form

$$\tau_{r\theta} = \sigma_0 (r_d/r)^2/\sqrt{3}. \tag{3.7}$$

From (3.3) and (3.7), it follows that

$$\sigma_e = \sigma_0 (r_d/r)^2. \tag{3.8}$$

Since σ_e is a known function of $\varepsilon_{\rm eq}$, the dependence of $\varepsilon_{\rm eq}$ on r and r_d is determined. We replace the section BC (see Fig. 1) by a straight line. Now

$$\sigma_e = \sigma_0 + \frac{\sigma_s - \sigma_0}{\varepsilon_s - \varepsilon_0} (\varepsilon_{\rm eq} - \varepsilon_0). \tag{3.9}$$

From (3.8) and (3.9), we find that

$$\varepsilon_{\rm eq} = \varepsilon_0 + (\varepsilon_s - \varepsilon_0)[(r_d/r)^2 - 1]/(p - 1), \qquad p = \sigma_s/\sigma_0. \tag{3.10}$$

Substituting (3.10) into (3.5) and making allowance for (3.2), we obtain

$$\frac{\partial (v/r)}{\partial r} = \frac{2\sqrt{3}(\varepsilon_s - \varepsilon_0) r_d v_n}{(p - 1) r^3}.$$

Hence,

$$v = -\frac{\sqrt{3}(\varepsilon_s - \varepsilon_0)v_n}{p-1}\frac{r_d}{r} + v_0\frac{r}{r_d}, \tag{3.11}$$

where v_0 is an arbitrary function of r_d. It follows from (3.1) and (3.11) that $v_0 = \sqrt{3}\,v_n(\varepsilon_s - p\varepsilon_0)/(p-1)$ and, hence,

$$v = \frac{\sqrt{3}\,v_n}{p-1}\,[(\varepsilon_s - p\varepsilon_0)(r/r_d) - (\varepsilon_s - \varepsilon_0)(r_d/r)]. \tag{3.12}$$

Here we have taken into account the condition $v < 0$, since it follows from (3.7) that $\tau_{r\theta} > 0$. From (3.2) and the boundary condition $v_\theta = -\omega R$, for $r = R$ we find that

$$v_n = \frac{dr_d}{dt} = -\frac{\omega(p-1)R}{\sqrt{3}\,[(\varepsilon_s - p\varepsilon_0)(R/r_d) - (\varepsilon_s - \varepsilon_0)(r_d/R)]}.$$

Then, Eq. (3.12) becomes

$$v = -\frac{\omega R[(\varepsilon_s - p\varepsilon_0)(r/r_d) - (\varepsilon_s - \varepsilon_0)(r_d/r)]}{(\varepsilon_s - p\varepsilon_0)(R/r_d) - (\varepsilon_s - \varepsilon_0)(r_d/R)}. \tag{3.13}$$

Solution (3.7) and (3.13) is valid in the plastic region $R \leqslant r \leqslant r_d$ with the initial condition $r_d = R$. Moreover, the solution fails when $\varepsilon_{\rm eq} = \varepsilon_s$. In this case, condition (1.3) holds for $r = R$, and the subsequent strain is localized on the velocity-discontinuity surface. In contrast to the surface $r = r_d$, this surface corresponds to condition (1.3); therefore, no additional restrictions are imposed on the velocity jump in this case. It follows from (3.10) that this stage of deformation begins when $r_d = Rp^{1/2}$.

Thus, the above solution implies that the material is in the initial state for $r \geqslant Rp^{1/2}$; for $R \leqslant r \leqslant Rp^{1/2}$, the material hardens, and the yield-point distribution over the radius is determined by Eqs. (3.9) and (3.10) for $r_d = Rp^{1/2}$.

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